ON THE THEORY OF OPTIMUM CONTROL

(K TEORII OPTIMAL'NOGO REGULIROVANIIA)

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Pontriagin and his students [1] have considered the general problem on optimal control and have derived the "principle of maximum". Rozonoer [2] proved this principle in a different way, established the connection between the method of dynamic programming of R. Bellman and Pontriagin's principle of maximum, and showed the analogy between these equations and the equations of analytical mechanics (Hamilton's equations and the Hamilton-Jacobi equations).

In the present work there is obtained a formula for the increment of the functional by a different method. It is shown that the problem of optimum control can be solved by the variational method with the aid of Lagrange multipliers. An explanation is given of the analogy between the equations of optimum control and the Lagrange equations in analytical mechanics. Some special cases are considered.

1. Statement of the problem. We shall consider the system of differential equations

$$\dot{x}_i = f_i(x_1, \ldots, x_n;\ u_1, \ldots, u_r;\ t) \qquad (i = 1, \ldots, n) \tag{1.1}$$

which describes the regulatory process of an automatic control system. Here $x_1(t), \ldots, x_n(t)$ are parameters of the control object, $u_1(t), \ldots,$ $u_r(t)$ are the positions of the regulating organs.

It is assumed that the functions f_i are continuous, bounded for all arguments and have continuous first-order partial derivatives

$$\partial f_i/\partial x_s \quad (i, s = 1, \ldots, n), \qquad \partial f_i/\partial u_k \quad \begin{pmatrix} i = 1, \ldots, n \\ k = 1, \ldots, r \end{pmatrix}$$

It is also assumed that $u_1, \ldots, u_r$ are piece-wise continuous and satisfy the inequalities

$$g_j(u_1, \ldots, u_r) \leqslant 0 \qquad (j = 1, \ldots, m) \tag{1.2}$$

In the sequel we shall refer to $u_1, \ldots, u_r$ as the "admissible controls".

Let us assume that at time t_0 the system is at the point $x^\circ = (x_1^\circ, \ldots, x_n^\circ)$ of the phase space. In [2] it was shown that the problem of optimum control can be reduced to the consideration of the system (1.1) (we assume that the new variables have already been intorudced in (1.1)), in which one has to select from the admissible controls which lead the system (1.1) from the point $x(t_0) = x^\circ$, the $u_1(t), \ldots, u_r(t)$ in such a way that at the given instant of time $t = T$ the sum

$$S = c_1 x_1(T) + \ldots + c_n x_n(T) \tag{1.3}$$

will take on a minimum (or maximum) value. Here, the c_i are certain constants.

2. Case when the trajectory has a free right end. We shall consider the case when no conditions are imposed on $x_1, \ldots, x_n$ when $t = T$.

Let $u_1, \ldots, u_r$ be optimum controls, i.e. they impart to the functional $S(T)$ (1.3) a minimum (or maximum) value. From (1.1) we have

$$\delta \dot{x}_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \delta x_j + \sum_{k=1}^{r} \frac{\partial f_i}{\partial u_k} \delta u_k + \varepsilon_i \qquad (i = 1, \ldots, n) \tag{2.1}$$

Here ϵ_i is an increment of the second or higher order. Multiplying the terms on both sides of this equation by $\lambda_i(t)$, we obtain

$$\lambda_i \delta \dot{x}_i = \lambda_i \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \delta x_j + \lambda_i \sum_{k=1}^{r} \frac{\partial f_i}{\partial u_k} \delta u_k + \lambda_i \varepsilon_i \qquad (i = 1, \ldots, n) \tag{2.2}$$

Next let us integrate both parts of (2.2) from t_0 to T. For the left-hand side we find

$$\int_{t_0}^{T} \lambda_i \delta \dot{x}_i \, dt = \lambda_i \delta x_i \Big|_{t_0}^{T} - \int_{t_0}^{T} \dot{\lambda}_i \, \delta x_i \, dt \tag{2.3}$$

From the condition

$$x_i(t_0) = x_i^\circ \qquad (i = 1, \ldots, n) \tag{2.4}$$

it follows that

$$\delta x_i(t_0) = 0 \qquad (i = 1, \ldots, n) \tag{2.5}$$

Furthermore, let us set

$$\lambda_i(T) = -c_i \qquad (i = 1, \ldots, n) \tag{2.6}$$

From this it follows that

$$\lambda_i \delta x_i \Big|_{t_0}^{T} = -c_i \delta x_i(T)$$

In accordance with these conditions we find that after integrating (2.2) we obtain

$$-c_i \delta x_i(T) = \int_{t_0}^{T} \left\{ \left[\lambda_i \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \delta x_j + \dot{\lambda}_i \delta x_i + \lambda_i \sum_{k=1}^{r} \frac{\partial f_i}{\partial u_k} \delta u_k \right] + \lambda_i \varepsilon_i \right\} dt \qquad (i = 1, \ldots, n) \tag{2.7}$$

Finally, carrying out the summation for i in (2.7), we obtain an expression for the increment of the functional (1.3) when $t = T$:

$$\Delta S(T) = \sum_{i=1}^{n} c_i \delta x_i(T) = -\int_{t_0}^{T} \left\{ \left[\sum_{i=1}^{n} \left(\dot{\lambda}_i + \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \right) \delta x_i + \right.\right.$$
$$\left.\left. + \sum_{k=1}^{r} \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial u_k} \delta u_k \right] + \sum_{i=1}^{n} \lambda_i \varepsilon_i \right\} dt \tag{2.8}$$

The linear part of Equation (2.8) is the variation of the functional when $t = T$, i.e.

$$\delta S(T) = -\int_{t_0}^{T} \left[\sum_{i=1}^{n} \left(\dot{\lambda}_i + \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \right) \delta x_i + \sum_{k=1}^{r} \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial u_k} \delta u_k \right] dt \tag{2.9}$$

If for the controls $u_1, \ldots, u_r$ the functional S has a minimum (or maximum) value when $t = T$, then the variation of the functional S will vanish when $t = T$, i.e. $\delta S(T) = 0$. From this it follows that the right-hand side of Equation (2.9) must be equal to zero.

The multipliers $\lambda_i(t)$ are selected so that

$$\dot{\lambda}_i + \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} = 0, \quad \text{or} \quad \dot{\lambda}_i = -\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \qquad (i = 1, \ldots, n) \tag{2.10}$$

Here one has to take into account the boundary conditions (2.6).

Furthermore, because of the independence of the variations $\delta u_1, \ldots, \delta u_r$, the right-hand side of (2.6) has to be zero, and in addition to the conditions (2.10) one has to have the conditions

$$\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial u_k} = 0 \qquad (k = 1, \ldots, r) \tag{2.11}$$

Thus, the set of equations (2.10), (2.6), (2.11), (1.1) and (2.4) form the system of equations of the problem under consideration.

Let us introduce the function

$$H = \lambda_1 f_1 + \ldots + \lambda_n f_n \tag{2.12}$$

Then the indicated system reduces to a system of Hamilton's equations

$$\dot{x}_i = \partial H / \partial \lambda_i, \qquad x_i(t_0) = x_i^{\circ} \qquad (i = 1, \ldots, n) \tag{2.13}$$

$$\dot{\lambda}_i = -\partial H / \partial x_i, \qquad \lambda_i(T) = -c_i \qquad (i = 1, \ldots, n) \tag{2.14}$$

$$\partial H / \partial u_k = 0 \qquad (k = 1, \ldots, r) \tag{2.15}$$

The condition (2.15) indicates that under optimum control $u_1, \ldots, u_r$ the function H will be an extremum.

From what has been said, it follows that the problem on optimum control can be solved by the method of Lagrange multipliers $\lambda_i(t)$. In fact, the problem can be reduced to the determination of the extremum of the integral

$$S = \int_{t_0}^{T} \sum_{i=1}^{n} c_i \dot{x}_i \, dt \tag{2.16}$$

under the conditions

$$\dot{x}_i - f_i(x_1, \ldots, x_n;\ u_1, \ldots, u_r;\ t) = 0 \qquad (i = 1, \ldots, n) \tag{2.17}$$

For the solution of this problem we construct a new function

$$L = \sum_{i=1}^{n} c_i \dot{x}_i + \sum_{i=1}^{n} \lambda_i (\dot{x}_i - f_i) \tag{2.18}$$

If the integral (2.16) takes on an extremal value for $u_1, \ldots, u_r$ for the corresponding $x_1, \ldots, x_n$, then by Lagrange's method

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0 \qquad (i = 1, \ldots, n) \tag{2.19}$$

$$\frac{\partial L}{\partial u_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}_k} = 0 \qquad (k = 1, \ldots, r) \tag{2.20}$$

Furthermore, Equations (1.1) and (2.17) can be written in the form

$$\frac{\partial L}{\partial \lambda_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}_i} = 0 \qquad (i = 1, \ldots, n) \tag{2.21}$$

Equations (2.19) and (2.20) are nothing more than Equations (2.10) and (2.11) or (2.14) and (2.15).

It should be noted here that in the application of Lagrange's method one must carefully determine the boundary conditions (2.6) for the differential equations (2.19).

Equations (2.19), (2.20) and (2.21) have the form of Lagrange's equations in analytical mechanics. Furthermore, between the functions H and L there exists the relation

$$H = -L + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i \tag{2.22}$$

3. Some other cases. I. We impose certain restrictions on the $x_i(t)$ $(i = 1, \ldots, n)$ when $t = T$.

1) *First case*. When $t = T$, the functions $x_i(T)$ $(i = 1, \ldots, n)$ can be subjected to the condition $F(x_1, \ldots, x_n) \leqslant 0$. Here we shall confine ourselves to the consideration of the case

$$F(x_1, \ldots, x_n) = 0 \qquad \text{when } t = T \tag{3.1}$$

In order that $\delta S(T) = 0$, we have from (2.9) that

$$\sum_{i=1}^{n}\left(\dot{\lambda}_i + \sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_i}\right)\delta x_i + \sum_{k=1}^{r}\sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial u_k}\delta u_k = 0 \quad \text{when } t_0 \leqslant t \leqslant T \tag{3.2}$$

The condition (3,1) can be considered as the new equation of constraint. If one now assumes the existence of the derivatives $\partial F/\partial x_i (i = 1, \ldots, n)$, it follows from the first variation of the function $F(x_1, \ldots, x_n)$ for $\delta x_i(T)$ $(i = 1, \ldots, n)$ that

$$\frac{\partial F}{\partial x_1}\delta x_1 + \ldots + \frac{\partial F}{\partial x_n}\delta x_n = 0 \tag{3.3}$$

This is the auxiliary condition on the δx_i $(i = 1, \ldots, n)$ in Equations (3.2). Not all of the $\partial F/\partial x_i$ $(i = 1, \ldots, n)$ are zero in (3.3), otherwise the function $F(x_1, \ldots, x_n)$ would not contain a single one of the variables $x_1, \ldots, x_n$.

Suppose, for example, that $\partial F/\partial x_n$ is not zero. Then it follows from (3.3) that

$$\delta x_n = -\left(\frac{\partial F}{\partial x_1}\delta x_1 + \ldots + \frac{\partial F}{\partial x_{n-1}}\delta x_{n-1}\right) \Big/ \frac{\partial F}{\partial x_n}$$

Substituting this expression into the first sum of Equation (3.2), we obtain

$$\sum_{i=1}^{n-1}\left[\dot{\lambda}_i + \sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_i}\right]\delta x_i +$$

$$+\left[\dot{\lambda}_n + \sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_n}\right]\left[-\left(\frac{\partial F}{\partial x_1}\delta x_1 + \ldots + \frac{\partial F}{\partial x_{n-1}}\delta x_{n-1}\right)\Big/\frac{\partial F}{\partial x_n}\right]$$

$$= \sum_{i=1}^{n-1}\left[\dot{\lambda}_i + \sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_i} - \frac{\partial F}{\partial x_i}\left(\dot{\lambda}_n + \sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_n}\right)\Big/\frac{\partial F}{\partial x_n}\right]\delta x_i$$

Let us select the $\lambda_n, \lambda_1, \ldots, \lambda_{n-1}$ so that

$$\dot{\lambda}_n + \sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_n} = \frac{\partial F}{\partial x_n}, \quad \dot{\lambda}_i + \sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_i} = \frac{\partial F}{\partial x_i} \qquad (i = 1, \ldots, n-1) \quad (3.4)$$

or, unifying the notation,

$$\dot{\lambda}_i = -\sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial x_i} + \frac{\partial F}{\partial x_i} \qquad (i = 1, \ldots, n) \qquad (3.5)$$

Then, if the relations (3.5) and (2.11) are both satisfied

$$\sum_{j=1}^{n}\lambda_j\frac{\partial f_j}{\partial u_k} = 0 \qquad (k = 1, \ldots, n)$$

$\delta S(T) = 0$ in accordance with (3.2) and (2.9). In the case under consideration one obtains Equations (3.5) in place of (2.10). The boundary conditions for (3.5) are taken, as before, in the form (2.6). If we introduce the function

$$H^\circ = \sum_{i=1}^{n}\lambda_i f_i - F \qquad (3.6)$$

we then obtain the canonical form of the equations

$$\dot{x}_i = \frac{\partial H^\circ}{\partial \lambda_i}, \quad \dot{\lambda}_i = -\frac{\partial H^\circ}{\partial x_i} \quad (i = 1, \ldots, n), \qquad \frac{\partial H^\circ}{\partial u_k} = 0 \quad (k=1, \ldots, r) \quad (3.7)$$

In this case the problem can be solved by Lagrange's method. We construct the function

$$\Phi = \sum_{i=1}^{n} c_i \dot{x}_i + \sum_{i=1}^{n} \lambda_i (\dot{x}_i - f_i) + \lambda_{n+1} F \tag{3.8}$$

in place of (2.18). From the equations

$$\frac{\partial \Phi}{\partial x_i} - \frac{d}{dt}\frac{\partial \Phi}{\partial \dot{x}_i} = 0 \quad (i = 1, \ldots, n), \qquad \frac{\partial \Phi}{\partial u_k} - \frac{d}{dt}\frac{\partial \Phi}{\partial \dot{u}_k} = 0 \quad (k = 1, \ldots, r) \tag{3.9}$$

we obtain

$$\dot{\lambda}_i = -\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} + \lambda_{n+1}\frac{\partial F}{\partial x_i} \qquad (i = 1, \ldots, n) \tag{3.10}$$

and also Equations (2.11).

Let us now consider the multiplier λ_{n+1}. The nonhomogeneous equations for the $\lambda_i (i = 1, \ldots, n)$ are given by the differential equations (3.10). Their particular solutions u_i have the form

$$u_i = \sum_{j=1}^{n} \lambda_i^{(j)}(t) \int_{t_0}^{T} \frac{1}{D(\tau)} \sum_{l=1}^{t} D_{lj}(\tau)\, \lambda_{n+1} \frac{\partial F}{\partial x_l}\, d\tau \qquad (D = \det \| \lambda_S^{(l)} \|) \tag{3.11}$$

Here, the $\lambda_1^{(l)}, \ldots, \lambda_n^{(l)}$ are a fundamental system, and D_{ij} is the minor with the proper sign of the element $\lambda_l^{(j)}$ in the determinant D. From (3.11) it can be seen that one can choose an arbitrary constant for a particular solution of λ_{n+1}. Let

$$\lambda_{n+1} = 1 \tag{3.12}$$

Then Equations (3.10) will be of the same form as (3.5). Furthermore, in Expression (3.8) of the function Φ one should also set $\lambda_{n+1} = 1$. One can solve the problem in an analogous manner if there are given several restrictions (3.1), i.e. if

$$F_S(x_1, \ldots, x_n) = 0 \qquad (S = 1, \ldots, m,\ m < n) \tag{3.13}$$

2) *Second case*. Let us suppose that when $t = T$ all the $x_i(T)$ $(i = 1, \ldots, n - 1)$ are fixed, while for $x_n(T)$ one is to find the minimum (or maximum) value. For example, one may be required to find the minimum transient process for some control system.

In this case one has to consider the boundary conditions of Euler's equation. When $t = T$, all the $x_i(T)$ $(i = 1, \ldots, n - 1)$ are fixed. Therefore, $\delta x_i(T) = 0$ $(i = 1, \ldots, n - 1)$. Hence, it is impossible to determine the $\lambda_i(T)$ $(i = 1, \ldots, n - 1)$ in this case; we have only

$$\lambda_n(T) = -1 \tag{3.14}$$

We thus obtain the following differential equations and boundary conditions:

$$\dot{x}_i = f_i(x_1, \ldots, x_n;\ u_1, \ldots, u_r;\ t) \qquad (i = 1, \ldots, n) \tag{3.15}$$

$$\dot{\lambda}_i = -\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} \quad (i = 1, \ldots, n) \qquad \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial u_k} = 0 \quad (k = 1, \ldots, r) \tag{3.16}$$

$$x_i(t_0) = x_i^{\circ} \quad (i = 1, \ldots, n) \qquad x_i(T) = x_i^{1} \quad (i = 1, \ldots, n-1)$$

Here, $x_i^{1}(i = 1, \ldots, n-1)$ are fixed values.

There exist, as yet, no general methods for solving these differential equations; in some investigations there are given solutions of several linear problems treated by various methods.

II. We shall derive one relation which is useful for solving some linear systems.

1) For the system

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)\, x_j + \sum_{k=1}^{r} b_{ik}(t)\, u_k \qquad (i = 1, \ldots, n) \tag{3.17}$$

we have

$$\delta\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)\, \delta x_j + \sum_{k=1}^{r} b_{ik}(t)\, \delta u_k \qquad (i = 1, \ldots, n) \tag{3.18}$$

Multiplying this equation by $\lambda_i(t)$ and integrating the result from t_0 to T, we obtain

$$\Delta S = \sum_{i=1}^{n} c_i\, \delta x_i(T) = -\int_{t_0}^{T} \Big[\sum_{i=1}^{n} \Big(\dot{\lambda}_i + \sum_{j=1}^{n} \lambda_j a_{ji}\Big) \delta x_i + \sum_{k=1}^{r} \sum_{j=1}^{n} \lambda_j b_{jk} \delta u_k\Big] dt \tag{3.19}$$

As is known, in order that the functional (1.3) have a minimum value, it is necessary that $\Delta S \geqslant 0$. Let us choose the $\lambda_i(t)$ so that

$$\dot{\lambda}_i = -\sum_{j=1}^{n} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{3.20}$$

Then, in order that the condition $\Delta S \geqslant 0$ be satisfied, it is necessary that

$$\sum_{k=1}^{r} \sum_{j=1}^{n} \lambda_j b_{jk} \delta u_k \leqslant 0 \tag{3.21}$$

Let

$$f_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{k=1}^{r} b_{ik}u_k \qquad (i = 1, \ldots, n) \tag{3.22}$$

Then the condition (3.21) can be expressed in the form

$$\sum_{k=1}^{r} \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial u_k} \delta u_k \leqslant 0 \tag{3.23}$$

If one now introduces the function $H = \lambda_1 f_1 + \ldots + \lambda_n f_n$, then one obtains

$$\sum_{k=1}^{r} \frac{\partial H}{\partial u_k} \delta u_k \leqslant 0, \quad \text{or} \quad \Delta H \leqslant 0 \tag{3.24}$$

This means that under a control process which is optimal for the minimum (maximum) value of the functional S, the function H attains a maximum (minimum) value. If one solves the problem in this case by Lagrange's method, then one obtains again from the equations

$$\frac{\partial \Phi}{\partial x_i} - \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{x}_i} = 0 \qquad (i = 1, \ldots, n)$$

the system (3.20). But by

$$\frac{\partial \Phi}{\partial u_k} - \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{u}_k} = 0 \qquad (k = 1, \ldots, r)$$

we have

$$\partial H / \partial u_k = 0 \qquad (k = 1, \ldots, r) \tag{3.25}$$

This condition indicates that the optimum controls $u_1, \ldots, u_r$ give an extremal value to the function H_i, but it is impossible to determine what type of extremum it is, a maximum or a minimum.

2) The above discussion of the system (3.17) applies also to the linear system

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t) x_j + \varphi_i(u_1, \ldots, u_r) = f_i \qquad (i = 1, \ldots, n) \tag{3.26}$$

3) The systems (3.17) and (3.26) can be written in the general form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t) x_j + X_i \qquad (i = 1, \ldots, n) \tag{3.27}$$

Multiplying (3.27) and (3.20) by λ_i and x_i, respectively, and summing with respect to i, we obtain

$$\frac{d}{dt}\sum_{i=1}^{n}\lambda_i x_i - \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}\lambda_i x_j - a_{ji}\lambda_j x_i) = \sum_{i=1}^{n}\lambda_i X_i$$

The second term on the left-hand side vanishes, and we obtain [3]

$$\frac{d}{dt}\sum_{i=1}^{n}\lambda_i x_i = \sum_{i=1}^{n}\lambda_i X_i \tag{3.28}$$

Integrating from t_1 to t_2, we obtain

$$\sum_{i=1}^{n}\lambda_i x_i \Big|_{t_2} = \sum_{i=1}^{n}\lambda_i x_i \Big|_{t_1} + \int_{t_1}^{t_2}\Big(\sum_{i=1}^{n}\lambda_i X_i\Big)\,dt \tag{3.29}$$

One can use the relation repeatedly in solving specific problems if one selects different appropriate values for t_1 and t_2.

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